EXERCISE – III SUBJECTIVE QUESTIONS

1. Let $f(x) = \begin{cases} x^2 & x \ge 0 \\ ax & x < 0 \end{cases}$. Find real values of a such that f(x) is strictly monotonically increasing at x = 0.

2. (i) Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is a decreasing function for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(ii) Show that $f(x) = \frac{x}{\sqrt{1+x}} - \ln(1 + x)$ is an increasing function for $x > -1$.

3. If $f(x) = x^3 + (a - 1)x^2 + 2x + 1$ is strictly monotonically increasing for every $x \in R$ then find the range of values of 'a'

4. Find the intervals of monotonocity for the following functions.

(i) $\frac{x^4}{4} + \frac{x^3}{3} - 3x^2 + 5$

(ii) $\sin\frac{\pi}{x}$

(iii) $\log_3^2 x + \log_3^x$

5. Find the values of 'a' for which the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases for all real values of x.

6. Find the greatest & least value of

$f(x) = \sin^{-1}\frac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$.

7. If g(x) is monotonically increasing and f(x) is monotonically decreasing for $x \in R$ and if (gof) (x) is defined for $x \in R$, then prove that (gof)(x) will be monotonically decreasing function. Hence prove that (gof) (x + 1) < (gof) (x – 1)

8. Using monotonicity prove that

(i) $x < -\ln(1 - x) < x(1 - x)^{-1}$ for $0 < x < 1$

(ii) $\frac{x}{1-x^2} < \tan^{-1} x < x$ for every $x \ge 0$

9. Prove that inequality, $\frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$ for $0 < x_1 < x_2 < \frac{\pi}{2}$

10. For $x \in \left(0, \frac{\pi}{2}\right)$ identify which is greater $(2\sin x + \tan x)$ or $(3x)$. Hence find $\lim_{x\to 0}\left[\frac{3x}{2\sin x + \tan x}\right]$ where [∗] denote the greatest integer function.

11. Using monotonocity find range of the function $f(x) = \sqrt{x-1} + \sqrt{6-x}$.

12. If $f : [0, \infty) \to R$ is the function defined by $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then whether f(x) is injective or not.

13. Prove that

$e^x + \sqrt{1+e^{2x}} \ge (1 + x) + \sqrt{2+2x+x^2} \quad \forall\ x \in R$.

14. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall\ x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the intervals of monotonocity of g(x).

15. If $ax^2 + (b/x) \ge c$ for all positive x where $a > 0$ and $b > 0$ then show that $27ab^2 \ge 4c^3$.

16. Find the set of all values of the parameter 'a' for which the function
$f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for a $x \in R$.

17. Find the set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a + 2)x^2 + (a - 1)x + 2$ possess a negative point of inflection.

18. Find which of the two is larger $ln\ (1 + x)$ or $\frac{\tan^{-1} x}{1+x}$, $x \geq 0$.

19. Using monotonicity prove that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $x \in (0, \pi/2)$

20. Let $f(x) = \begin{cases} \max(x, x^2) & x \geq 0 \\ \min(x, x^2 - 2) & x < 0 \end{cases}$. Draw the graph of f(x) and hence comment on the nature of monotonic behaviour at x = −1, 0, 1.

21. Find the values of 'a' for which the function

$f(x) = \sin x - a \sin 2x - \frac{1}{3} \sin 3x + 2ax$ increases throughout the number line.

22. Prove the following inequalities

(i) $1 + x^2 > (x \sin x + \cos x)$ for $x \in [0, \infty)$

(ii) $\sin x - \sin 2x \leq 2x$ for all $x \in \left[0, \frac{\pi}{3}\right]$

(iii) $\frac{x^2}{2} + 2x + 3 \geq (3 - x)e^x$ for all $x \geq 0$

23. Prove that $0 < x \sin x - \frac{\sin^2 x}{2} < \frac{1}{2} (\pi - 1)$ for $0 < x < \frac{\pi}{2}$.

24. Find the interval to which b may belong so that the function $f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right) x^3 + 5x + \sqrt{6}$ is increasing at every points of its domain.

25. Show that $x^2 > (1 + x)\ [ln\ (1 + x)]^2\ \forall\ x > 0$.

26. Find the intervals of monotonocity for the following functions & represent your solution set on the number line.
Also plot the graph in each case.

(a) $f(x) = 2.\ e^{x^2 - 4x}$

(b) $f(x) = e^x/x$

(c) $f(x) = x^2 e^{-x}$

(d) $f(x) = 2x^2 - ln\ |x|$

27. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$.

28. Find the intervals of monotonocity of the function
(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$

(b) $g(x) = 2 \sin x + \cos 2x$ in $x \in [0, 2\pi]$

29. Let $f(x) = x^3 - x^2 + x + 1$ and
$g(x) = \begin{cases} \max\{f(t) : 0 \leq t \leq x\} & , 0 \leq x \leq 1 \\ 3 - x & , 1 < x \leq 2 \end{cases}$
Discuss the continuity & differentiability of g(x) is in the interval (0, 2)

30. Find the greatest & the least values of the following functions in the given interval if they exist.
(a) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

(b) $y = x^5 - 5x^4 + 5x^3 + 1$ in [−1, 2]

31. If $f(x) = \left(\frac{a^2 - 1}{3}\right) x^3 + (a - 1) x^2 + 2x + 1$ is monotonic increasing for every $x \in R$ then find the range of values of 'a'.

32. Find the range of values of 'a' for which the function $f(x) = x^3 + (2a + 3) x^2 + 3(2a + 1) x + 5$ is monotonic in R. Hence find the set of values of 'a' for which f(x) in invertible.

33. Find the value of x > 1 for which the function

$F(x) = \int_x^{x^2} \frac{1}{t} \ln\left(\frac{t - 1}{32}\right) dt$ is increasing and decreasing.

34. If $f(x) = 2e^{x} - ae^{-x} + (2a + 1)x - 3$ monotonically increases for every $x \in R$ then find the range of values of 'a'.

35. Construct the graph of the function

$f(x) = -\left|\frac{x^2-9}{x+3} - x + \frac{2}{x-1}\right|$ and comment upon the following

(a) Range of the function,

(b) Intervals of monotonocity,

(c) Point(s) where f is continuous but not differentiable,

(d) Point(s) where f fails to be continuous and nature of discontinuity.

(e) Gradient of the curve where f crosses the axis of y.

36. Prove that, $x^2 - 1 > 2x \ln x > 4x(x - 1) - 2 \ln x$ for $x > 1$.

37. Prove that $\tan^2 x + 6 \ln \sec x + 2 \cos x + 4 > 6 \sec x$ for $x \in \left(\frac{3\pi}{2}, 2\pi\right)$.

38. Find the set of values of x for which the inequality $\ln(1 + x) > x/(1 + x)$ is valid.

Sol.

39. If b > a, find the minimum value of $|(x - a)^3| + |(x - b)^3|$, $x \in R$.